Local Rings and Completions

Williams College SMALL REU
Commutative Algebra Group

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Local rings are unusual, but we can make any Noetherian ring into a local ring using a proccess called *localization*. A ring R localized at a prime ideal P is denoted R_P.

Let (R, M) be a local ring.

Definition

The *M-adic metric* on R is given by

$$d(x,y) = \begin{cases} \frac{1}{2^n} & n = \max\{k \mid x - y \in M^k\} \text{ if it exists} \\ 0 & \text{otherwise} \end{cases}$$

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The *completion* of R, denoted by $\widehat{R}$, is the completion of R as a metric space with respect to the M-adic metric.

$\widehat{R}$ is equipped with a natural ring structure.

Example: $\widehat{\mathbb{Q}[x]}_{(x)} = \mathbb{Q}[[x]]$.

Motivation

Theorem (Cohen Structure Theorem)

If T is a complete local ring containing a field, then $T \cong K[[x_1, \ldots, x_n]]/I$ for some field K and ideal I of $K[[x_1, \ldots, x_n]]$.

We understand complete rings very well because of the Cohen structure theorem. If we understand the relationship between a ring and its completion, we can learn about an arbitrary local ring by passing to its completion.

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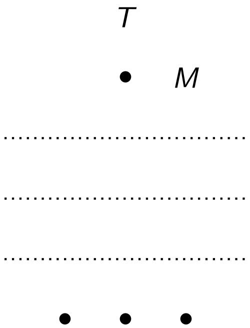
If the generic formal fiber of R has a single maximal element, then we say R has a *local* generic formal fiber.

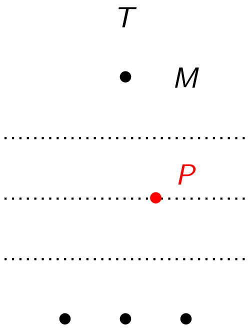
Previous Results

Theorem (P. Charters and S. Loepp, 2004)

Let (T, M) be a complete local ring of characteristic 0 and P a prime ideal of T. Then T is the completion of a local excellent domain A posessing a local generic formal fiber with maximal ideal P if and only if T is a field and $P = (0)$ or the following conditions hold:

1. $P \neq M$
2. *P contains all zero divisors of T and no nonzero integers of T,*
3. *T_P is a regular local ring.*


T
M


T
M
P


A
$M \cap A$
T
M
P


A
M ∩ A
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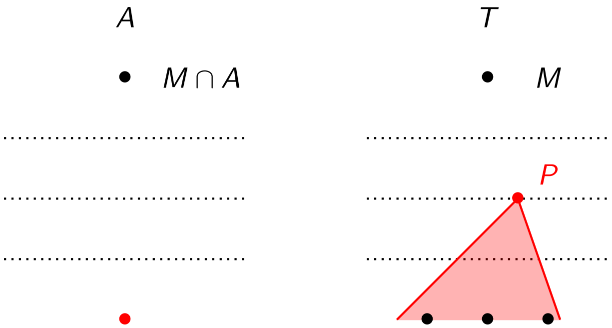
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"It has been generally agreed that 'excellent' Noetherian rings should behave similarly to the rings found in algebraic geometry, specifically, rings of the form

$$A = K[x_1, \ldots, x_n]/I$$

where A has finite type over a field K."
(C. Rotthaus, *Excellent Rings, Henselian Rings, and the Approximation Property*, Rocky Mountain J. Math 1997)

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As Charters and Loepp noted, "this proof fails if the characteristic of T is $p > 0$, as the ring we construct may not have a geometrically regular generic formal fiber."

That is, we need to construct A so that $T \otimes_A L$ is a regular ring for every finite extension L of K, where K is the quotient field of A.

Definition

A local ring (R, M) is a *regular local ring* if the minimal number of generators of M is equal to the length of the longest chain of prime ideals

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Definition

A Noetherian ring R is *regular* if the localization of R at every prime ideal is a regular local ring.

Recall: A is a local integral domain with quotient field K, $\widehat{A} = T$, $P \in \operatorname{Spec} T$, and L is a finite extension of K.

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In characteristic 0, K has no non-trivial purely inseparable extensions, so we only need to check that $T \otimes_A K$ is regular. In fact, $T \otimes_A K \cong T_P$ so this is condition 3 of the Charters and Loepp theorem.

Theorem (P. Charters and S. Loepp, 2004)

Let (T, M) be a complete local ring of characteristic 0 and P a prime ideal of T. Then T is the completion of a local excellent domain A posessing a local generic formal fiber with maximal ideal P if and only if T is a field and $P = (0)$ or the following conditions hold:

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In non-zero characteristic, K can have non-trivial purely inseparable extensions, so it is much harder.

Results

Theorem (SMALL 2013 Comm. Alg.)

Let (T, M) be a complete local ring of characteristic p, P a prime ideal of T, and A a local domain with completion T and local generic formal fiber with maximal element P. Let K be the quotient field of A. Then for any finite purely inseparable field extension L of K,

$$T \otimes_A L \cong T_P[x_1, \ldots, x_r]/\langle x_1^{p^{n_1}} - k_1, \ldots, x_r^{p^{n_r}} - k_r \rangle$$

for some $n_i \in \mathbb{N}$ and $k_i \in K[x_1, \ldots, x_{i-1}]$.

Theorem (SMALL 2013 Comm. Alg.)

Let (R, M) be a regular local ring of characteristic p, and $k \in R$. Then $R[x]/\langle x^{p^n} - k\rangle$ is regular (in fact, regular local) if and only if $k + M^2$ is not a p^{th} power in R/M^2.

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This allows us to classify when $T \otimes_A K$ is geometrically regular (i.e. $T \otimes_A L$ is regular for every finite purely inseparable extension L of K).

Corollary (SMALL 2013 Comm. Alg.)

Let A be a local domain with completion $\widehat{A} = T$ and quotient field K. Then $T \otimes_A K$ is geometrically regular if and only if for every sequence $k_1 \in K, k_2 \in K[x_1], \ldots, k_n \in K[x_1, \ldots, x_{n-1}]$ such that k_i is not a p^{th} power in

$$K[x_1, \ldots, x_{i-1}]/\langle x^{p^{n_1}} - k_1, \ldots, x^{p^{n_{i-1}}} - k_{i-1}\rangle,$$

k_i is also not a p^{th} power in

$$(T_P[x_1, \ldots, x_{i-1}]/\langle x^{p^{n_1}} - k_1, \ldots, x^{p^{n_{i-1}}} - k_{i-1}\rangle)/M_i^2$$

where M_i is the maximal ideal of $T_P[x_1, \ldots, x_{i-1}]/\langle x^{p^{n_1}} - k_1, \ldots, x^{p^{n_{i-1}}} - k_{i-1}\rangle$.

Conjecture

Let (T, M) be a complete local ring of any characteristic and P a prime ideal of T. Then T is the completion of a local excellent domain A posessing a local generic formal fiber with maximal ideal P if and only if T is a field and $P = (0)$ or the following conditions hold:

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Any questions?